A high-order positivity-preserving single-stage single-step method for the ideal magnetohydrodynamic equations

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Ideal magnetohydrodynamic equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \mathcal{E}\\ \mathbf{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\otimes\mathbf{u} + (p+\frac{1}{2}\mathbf{B}^2)\mathbb{I} - \mathbf{B}\otimes\mathbf{B}\\ \mathbf{u}(\mathcal{E}+p+\frac{1}{2}\mathbf{B}^2) - \mathbf{B}(\mathbf{u}\cdot\mathbf{B})\\ \mathbf{u}\otimes\mathbf{B} - \mathbf{B}\otimes\mathbf{u}\end{bmatrix} = 0,$$

$$\nabla\cdot\mathbf{B} = 0$$

where ρ: mass density, $\rho\mathbf{u}$: momentum, $\mathcal{E}$: total energy, p: thermal pressure, $\|\cdot\|$: Euclidean vector norm, $\gamma = 5/3$: ideal gas constant,

equation of state: $\mathcal{E} = \frac{p}{\gamma-1} + \frac{\rho\|\mathbf{u}\|^2}{2} + \frac{\|\mathbf{B}\|^2}{2}$

Goal

Develop a building block for high-order finite difference MHD code with AMR

Keywords

- Single-stage, single-step
- PIF-WENO
- Unstaggered constrained transport
- Magnetic potential evolution equation treated as Hamilton-Jacobi equations
- Positivity-preserving limiter on fluxes

Roadmap

- Single-stage single-step constrained transport ✓
- Positivity-preserving ✓
- AMR COMING SOON
- IMEX method for non-ideal MHD COMING SOON
- Curvilinear mesh COMING SOON

Single-stage, single-step

Single-stage No intermediate stage to store
Single-step No history to store

Benefits in potential embedding to AMR:

- No intermediate stage ⇝ fewer communications
- Smaller effective stencil ⇝ fewer ghost points per communication

AMR

- Adaptive Mesh Refinement
- Efficient when complex features are localized
- Motivation and a next step of current work

PIF-WENO scheme

Hyperbolic conservation law $q_t + f(q)_x + g(q)_y = 0 \implies$
Picard-Integral-Formulation $q^{n+1} = q^n - \Delta t(F^n(x,y))_x - \Delta t(G^n(x,y))_y$
where $F^n(x,y) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(q(t,x,y))\,dt, \quad G^n(x,y) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} g(q(t,x,y))\,dt$

Showing 2D, works for any-D

A discretization (Taylor):

1. Taylor expansion in time + Cauchy-Kowalewski + central differencing

$$F^n(x,y) = \left(f + \frac{\Delta t}{2!}\frac{df}{dt} + \frac{\Delta t^2}{3!}\frac{d^2f}{dt^2}\right)\Bigg|_{t^n} + \mathcal{O}(\Delta t^3)$$

$$\approx f + \frac{\Delta t}{2!}\frac{\partial f}{\partial q}\left(-f_x - g_y\right) + \frac{\Delta t^2}{3!}\left(\frac{\partial^2 f}{\partial q^2}(f_x+g_y, f_x+g_y) - \frac{\partial f}{\partial q}\left(\frac{\partial^2 f}{\partial q^2}(q_x, f_x+g_y) + \frac{\partial f}{\partial q}(f_{xx}+g_{xy}) + \frac{\partial^2 g}{\partial q^2}(q_y, f_x+g_y) + \frac{\partial g}{\partial q}(f_{xy}+g_{yy})\right)\right)$$

$G^n(x,y)$ similar

2. $F^n_{x\,i,j} := \frac{\hat{F}^n_{i+1/2,j} - \hat{F}^n_{i-1/2,j}}{\Delta x}$, $G^n_{y\,i,j} := \frac{\hat{G}^n_{i,j+1/2} - \hat{G}^n_{i,j-1/2}}{\Delta y}$, where $\hat{F}$, $\hat{G}$ are fluxes **WENO reconstructed** from F^n, G^n
3. Update q

Constrained transport

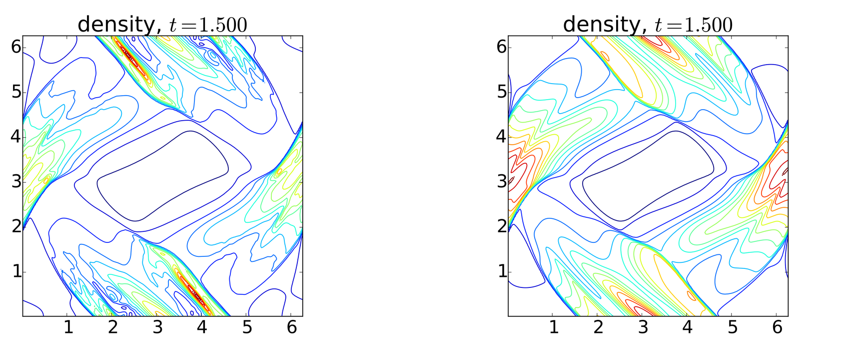
At each time step

1. PIF-WENO to evolve $(\rho^n, (\rho\mathbf{u})^n, \mathcal{E}^n, \mathbf{B}^n) \rightsquigarrow (\rho^{n+1}, (\rho\mathbf{u})^{n+1}, \mathcal{E}^{n+1}, \mathbf{B}^*)$
2. Evolve magnetic potential $\mathbf{A}^n \rightsquigarrow \mathbf{A}^{n+1}$
3. Correct $\mathbf{B}^{n+1} := \nabla\times\mathbf{A}^{n+1}$

Why bother correct B

Otherwise error in $\nabla\cdot\mathbf{B}$ accumulates, causing numerical instability

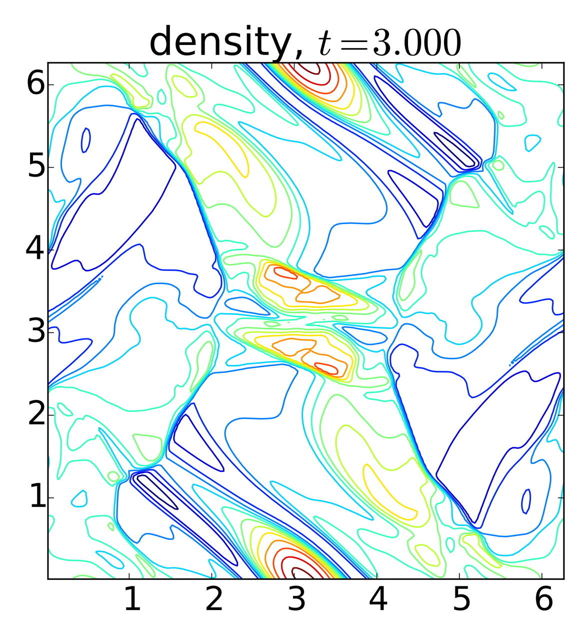
Orszag-Tang test



No constrained transport — With constrained transport

$\nabla\cdot\mathbf{B} = 0 \rightsquigarrow$ robustness

Orszag-Tang test



Unstaggered

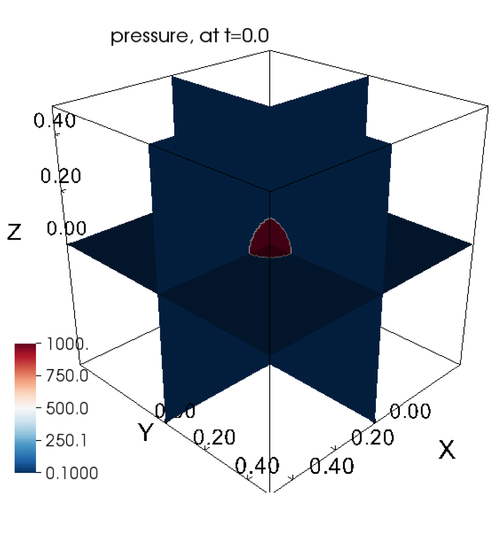
- $\mathbf{A}$ sits on same mesh as q
- Simplifies interpolations in future embedding to AMR

Evolution of A

- $\mathbf{A}_t + \begin{pmatrix}0 & -u^2 & -u^3\\ 0 & u^1 & 0\\ 0 & 0 & u^1\end{pmatrix}\mathbf{A}_x + \begin{pmatrix}u^2 & 0 & 0\\ -u^1 & 0 & -u^3\\ 0 & 0 & u^2\end{pmatrix}\mathbf{A}_y + \begin{pmatrix}u^3 & 0 & 0\\ 0 & u^3 & 0\\ -u^1 & -u^2 & 0\end{pmatrix}\mathbf{A}_z = 0$
- $\mathbf{A}^{n+1} \approx \mathbf{A}^n + \Delta t\mathbf{A}^n_t + \frac{\Delta t^2}{2!}\mathbf{A}^n_{tt} + \frac{\Delta t^3}{3!}\mathbf{A}^n_{ttt}$
  - $\mathbf{A}_t$: approximate by Lax-Friedrichs **numerical Hamiltonian**, with high-order reconstruction of $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$. Also, **artificial viscosity** in 3D to deal with weak hyperbolicity
  - $\mathbf{A}_{tt}, \mathbf{A}_{ttt}$: Cauchy-Kowalewski

This scheme controls oscillations in $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$, thus controls oscillations in $\mathbf{B}$

Positivity-preserving



- Low β (≈ 0.00025)
- Steep (1000 : 0.1) contrast in pressure

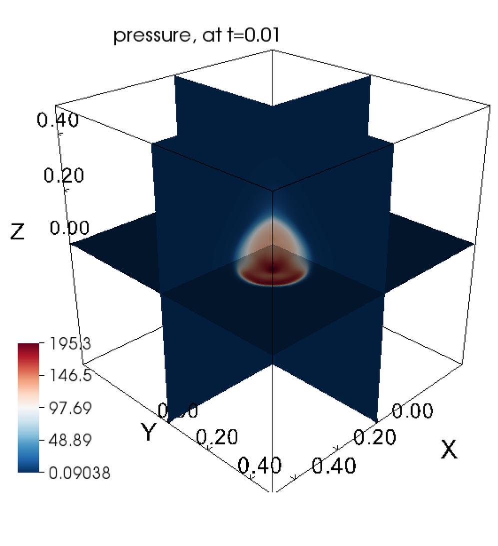
- Negative pressure in simulation (!!) ⇝
- Limit fluxes towards low-order (establishedly positivity-preserving) fluxes, e.g. in 2D, use limited fluxes

$\tilde{F}_{i+1/2,j} = \theta_{i+1/2,j}(\hat{F}_{i+1/2,j} - \hat{f}_{i+1/2,j}) + \hat{f}_{i+1/2,j}$

$\tilde{G}_{i,j+1/2} = \theta_{i,j+1/2}(\hat{G}_{i,j+1/2} - \hat{g}_{i,j+1/2}) + \hat{g}_{i,j+1/2}$

where $\hat{f}$, $\hat{g}$ are Lax-Friedrichs fluxes, and θ's $\in [0,1]$

- Retains positive density and pressure
- High-order when no risk of negativity



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